Geometry of an accelerated rotating disk

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Abstract

We analyze the geometry of a rotating disk with a tangential acceleration in the framework of the theory of Special Relativity, using the kinematic linear differential system that verifies the relative position vector of time-like curves in a Fermi reference. A numerical integration of these equations for a generic initial value problem is made up and the results are compared with those obtained in other works.

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1   Introduction

The geometry of a rotating disk has generated an enormous literature (see the recent review by Grøn [1] in which one of the basic problems treated is connected with the precise definition of the space representing the disk. In applications such as the study of the motion of a rotating disk, when gravitational effects are discarded, splits of the Minkowski spacetime naturally occurs. One of these splits considers the Minkowski spacetime together with a congruence defined by a timelike vector field. Then, following Cattaneo [2], a *spatial metric* $ds^{\perp}$ and a *standard time* dx^0 relative to the congruence can be introduced, so that quantities such as the ratio $ds^{\perp}/dx^0$ have a physical and operational meaning. The relationship of this congruence splitting and that defined by a foliation of spacetime by spacelike hypersurfaces has been shown for general curved spacetimes, e.g., in Bini *et al.* [3]. The "hypersurface point of view" for the rotating disk has been considered , among others, by Tartaglia [4] for a definition of "space" of a rotating disk and by the present authors, in a discussion of the Sagnac effect [5]. The "congruence point of view" has been recently used in the work by Rizzi & Ruggiero [6], for the study of the space geometry of rotating

platforms and, by Minguzzi [7], for the study of simultaneity in stationary extended frames.

In [6], it is shown that each element of the periphery of the disk, of a given proper length, is stretched during the acceleration period using for this the Grøn model [8], in which the motion of the disk is not Born-rigid in the acceleration period. In [8], this dilatation of length is discussed using a kinematical argument to calculate this change taking into account the asynchrony of the acceleration measured from the rotating frame.

In this work we study the evolution of the distance between any two nearby points on the periphery of an accelerated rotating disk, using the geometrical properties of the timelike congruences in the special relativistic spacetime associated to the disk. The motion is described from a Fermi reference field, solving numerically the differential equations of the separation or relative position vector of timelike curves. Note that we do not introduce any coordinate system in a reference field as in [9].

The paper is organized as follows. In Sec. 2 it is described the material system showing the properties of the congruences associated to a uniformly accelerated rotating disk. In Sec. 3 we construct a field of Fermi references on the world tube corresponding to the flow of the disk in the spacetime. Further, in this section, the evolution of the length of an arc of circumference is established as an initial value problem for the deviation of nearby timelike curves. In Sec. 4 this problem is numerically integrated and the results obtained are compared with those given by Grøn.

2 Description of the material system

Let us consider the usual system of Cartesian coordinates $\mathrm{x} = (x, y, z, t)$ defined on the Minkowskian space-time $\mathcal{M}$, and assume that at time $t = 0$ the coordinate origin O is on the center of a circular disk whose radius is R and its symmetry axis is z. Throughout this paper we restrict ourselves to the three-dimensional submanifold $\mathcal{T} \subset \mathcal{M}$ given by $z = 0$. The η–orthonormal reference field $\mathbf{e}_{\mathrm{a}} := \{\partial_t, \partial_x, \partial_y\}$ is defined on $\mathcal{T}$. Due to the cylindrical symmetry of the problem it will be useful to introduce a cylindrical coordinate system on $\mathcal{T}$ with origin at O, defined by

$$t = t, \quad x = r \cos \phi, \quad y = r \cos \phi. \tag{1}$$

These coordinate systems determine a reference frame $\boldsymbol{e}_a := \{\partial_t, \partial_r, \partial_\phi\}$ on $\mathcal{M}$, related to $\{\mathbf{e}_{\mathrm{a}}\}$ by the classical expressions

$$\boldsymbol{e}_0 = \mathbf{e}_0, \quad \boldsymbol{e}_1 = r \cos \phi \, \mathbf{e}_1 + r \sin \phi \, \mathbf{e}_2, \quad \boldsymbol{e}_2 = -r \sin \phi \, \mathbf{e}_1 + r \cos \phi \, \mathbf{e}_2. \tag{2}$$

In addition to the coordinate systems $\{\mathbf{x}\}$ and $\{\boldsymbol{x}\}$, we also use a convected coordinate system: $\{X\} : (T, R, \Phi)$, co-rotating with the disk, de-

fined in terms of $\{\boldsymbol{x}\}$ by the coordinate transformation:

$$T = t, \quad R = r, \quad \Phi = \phi - \phi(t), \tag{3}$$

where $\phi(t)$ is a smooth function of t. The corresponding reference frame field on $\mathcal{M}$ is given by $\boldsymbol{E}_A := \{\partial_T, \partial_R, \partial_\Phi\}$, which is related to $\{\boldsymbol{e}_a\}$ through the expressions

$$\boldsymbol{E}_0 = \boldsymbol{e}_0 + \dot{\phi}(t)\boldsymbol{e}_2, \quad \boldsymbol{E}_1 = \boldsymbol{e}_1, \quad \boldsymbol{E}_2 = \boldsymbol{e}_2. \tag{4}$$

In the Cartesian coordinates $\mathbf{x}$, the matrix representation of the Minkowskian metric is $\boldsymbol{\eta}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$, where $\eta_{ab} = \text{diag}\,(-1, 1, 1)$. In terms of convective coordinates $\{\boldsymbol{X}\}$ this metric takes the form $\boldsymbol{\eta}(\boldsymbol{E}_A, \boldsymbol{E}_A) = G_{AB}$, where G is the matrix

$$G = \begin{pmatrix} R^2\varpi(T)^2 - 1 & 0 & R^2\varpi(T) \\ 0 & 1 & 0 \\ R^2\varpi(T) & 0 & R^2 \end{pmatrix} \tag{5}$$

where $\varpi(T) := \omega + T\alpha$, $\omega := \dot{\phi}(t)$ is the coordinate angular speed and $\alpha := \ddot{\phi}(t)$ the coordinate angular acceleration.

In the coordinate system $\{\mathbf{x}\}$ fixed in the space-time, the worldlines of points in the accelerated rotating disk are curves parametrized by the coordinate time

$$\mathbf{x}(t) = \Big(t, \, r_0 \cos\big(\phi_0 + \phi(t)\big), \, r_0 \sin\big(\phi_0 + \phi(t)\big) \Big). \tag{6}$$

Using convective coordinates these curves may be represented as $\boldsymbol{X}(T) = (T, R_0, \Phi_0)$, so that each worldline is identified by means of a pair (R_0, Φ_0). The tangent vector field to this flow is given in convective representation as

$$\dot{\boldsymbol{X}}(T) = \boldsymbol{E}_0, \tag{7}$$

where $\dot{\boldsymbol{X}}$ denotes the derivative of X with respect to T. Therefore, the tangent vector field to the flow, given by the space-time velocity

$$\boldsymbol{V} := (-G_{00}(X))^{-1/2}\boldsymbol{E}_0, \tag{8}$$

takes in the convective representation the expression

$$\boldsymbol{V} = \big((1 - R^2\varpi(T)^2)^{-1/2}, 0, 0\big). \tag{9}$$

For the vector field $\boldsymbol{V}(T)$ on the curve $\boldsymbol{X}(T)$ the proper acceleration A^I in convective coordinates is

$$A^I := \frac{DV^I}{ds} = \frac{dV^I}{ds} + \Gamma^I_{BC} V^B \frac{dX^C}{ds}, \tag{10}$$

where s denotes the proper time on the curve $\boldsymbol{X}(T)$.

3 Construction of a Fermi reference frame field

Now we will construct on a curve $\boldsymbol{X}(T) = (T, R, \Phi)$ a G–orthonormal reference frame field satisfying the Fermi propagation law. Given a G–orthonormal reference, $\boldsymbol{E}_{\hat{a}}(0)$, at an initial coordinate time $T = 0$, with $\boldsymbol{E}_{\hat{0}}(0) = \boldsymbol{V}(0)$, the relation between this reference and the comoving coordinate basis is

$$\boldsymbol{E}_{\hat{a}}(0) = E_{\hat{a}}^A(0)\, \boldsymbol{E}_A(X_0). \tag{11}$$

In the Fermi transport the evolution of the vector $\boldsymbol{E}_{\hat{0}}(0)$ is given by $\boldsymbol{V}(s)$. Thus, only the evolution of the spacelike vectors $\boldsymbol{E}_{\hat{\alpha}}$ of the reference frame must be determined. The absolute derivative of the field $\boldsymbol{E}_{\hat{\alpha}}$, satisfies the equations

$$\frac{DE_{\hat{\alpha}}^I}{ds} = V^I A^B G_{BC} E_{\hat{\alpha}}^C, \qquad \hat{\alpha} = 1, 2, \tag{12}$$

which can be expressed in terms of the coordinate time as the ordinary differential equation

$$\frac{dE_{\hat{\alpha}}^A}{dT} = \sqrt{-G_{00}}(-\Gamma_{BC}^A V^B + V^A A^B G_{BC}) E_{\hat{\alpha}}^C. \tag{13}$$

The integration of this differential system with the initial condition (11) gives the evolution of $\boldsymbol{E}_{\hat{a}}(T) = E_{\hat{a}}^A(T)\, \boldsymbol{E}_A(T)$, i.e., of the Fermi reference along the curve $\boldsymbol{X}(T)$.

The circumference of the rotating disk is identified with any of the circumferences $\mathcal{C}_T$ obtained by cutting the world tube of the disk with planes $t = \text{const}$ which are orthogonal to the worldline of the center of the disk. The construction of the quotient space $\mathcal{D}$ of $\mathcal{T}$ by the flow of time-like curves corresponding to $\mathcal{T}$ determines the material space associated to the disk. At each time T, a value of the line element is assigned to each circumference $\mathcal{C}_T$, using the metric induced by the Minkowskian metric on $\mathcal{T}$.

Let us denote by C the circumference of $\mathcal{D}$ and let $\tilde{P}$ and $\tilde{P}'$ be two infinitesimally neighboring points on C. Now, consider the timelike curves $\sigma(s)$ and $\sigma'(s)$ on $\mathcal{M}$ which are projected on the points $\tilde{P}$ and $\tilde{P}'$ respectively. In an arbitrary point P in $\sigma(s)$, the tangent space $T_P\mathcal{M}$ can be split as $T_P\mathcal{M} = \mathcal{H}_P \oplus \mathcal{V}_P$, where $\mathcal{V}_P$ represents the vector subspace whose elements are vectors parallel to the tangent vector $\boldsymbol{V}$ to $\sigma(s)$ at the point P, and $\mathcal{H}_P$ is the G–orthogonal complement to $\mathcal{V}_P$. Every vector $\boldsymbol{X} \in T_P\mathcal{M}$ can be projected on $\mathcal{H}_P$ using the projector $\boldsymbol{h} = \boldsymbol{1} + \boldsymbol{V} \otimes \boldsymbol{V}^\flat$, where $\boldsymbol{V}^\flat$ denotes the 1–form dual to $\boldsymbol{V}$. This projector corresponds to a metric $\boldsymbol{h}^\flat = \boldsymbol{G} + \boldsymbol{V}^\flat \otimes \boldsymbol{V}^\flat$ on $\mathcal{H}$. In each time s, the subspace $\mathcal{H}_P$ may be

identified with the space $T_{\tilde{P}}\mathcal{D}$. A further identification may be established between the space-like vectors of the Fermi reference $\{\boldsymbol{E}_{\hat{a}}(P;s)\}$

and the spacelike vectors of the co-rotating reference $\{\boldsymbol{E}_{\Xi}(\tilde{P})\}$ on $T_{\tilde{P}}\mathcal{D}$.

The measure of the distance between two $P_1 \in \sigma_1(s)$ and $P_2 \in \sigma_2(s)$ from a Fermi reference is carried out determining the length of the vector $\boldsymbol{S}$ of relative separation using the metric $\eta_{\hat{a}\hat{b}} = \mathrm{diag}\,(-1,1,1)$. On the other hand, in the material description on the quotient space $\mathcal{D}$, the relative position of the points $\tilde{P}_1$ and $\tilde{P}_1$ is constant, however the metric $\boldsymbol{h}(s)$ depends on time.

Following [10], we determine the rate of change of the separation of the points P_1 and P_2 as measured in $\mathcal{H}_P$, i.e., the rate of change of the relative position vector. Let $\lambda(\Phi)$ be a parametrization of the circumference $\mathcal{C}_0$ at the time $T = 0$. The tangent vector field to $\mathcal{C}_0$ is $\boldsymbol{S} = \partial_{\Phi}|_{\lambda(\Phi)}$. Consider the family of curves $\lambda(\Phi, s)$, obtained moving each point $\lambda(\Phi)$ a distance s on the corresponding integral curve of the flow of $\boldsymbol{V}$. Now, defining the connecting vector field $\boldsymbol{S} := \partial_{\Phi}|_{\lambda(\Phi,s)}$ such that the Lie derivative $\mathcal{L}_{\boldsymbol{V}}\boldsymbol{S}$ is zero, one obtains that the convective representation of $\boldsymbol{S}$ must satisfy

$$\frac{DS^A}{ds} = V^A_{;B}\, S^B. \tag{14}$$

Next, defining the separation or relative position vector between the points $\tilde{P}_1, \tilde{P}_2 \in \mathcal{D}$, measured in the convective frame, as

$$Y^A := h^A_{\ B} S^B, \tag{15}$$

one obtains that the evolution of this vector given by (14) is $\left(DY^A/ds\right)^{\perp} = V^A_{;B} Y^B$, where $\perp: T_P\mathcal{M} \to \mathcal{H}$, $\boldsymbol{X}^{\perp} := \boldsymbol{h} \cdot \boldsymbol{X}$. Then, using the definition of the Fermi derivative of a vector field $\boldsymbol{X}$ along a curve $\sigma(s)$:

$$\frac{D_F X^A}{ds} := \frac{DX^A}{ds} - (X^B A_B)V^A + (X^B V_B)V^A, \tag{16}$$

for which the relation $(D\boldsymbol{X}^{\perp}/ds)^{\perp} = D_F\boldsymbol{X}^{\perp}/ds$ holds, one can verify that (16) is equivalent to $D_F Y^A/ds = V^A_{;B} Y^B$. Now, choosing a Fermi reference on the base curve $\sigma_1(s)$, (14) can be written as

$$\frac{dY^{\hat{\alpha}}}{ds} = V^{\hat{\alpha}}_{;\hat{\beta}} Y^{\hat{\beta}}, \tag{17}$$

where $V^{\hat{\alpha}}_{;\hat{\beta}}$ is the projection of the covariant derivative $\nabla \boldsymbol{V}$ on $\mathcal{H}_P \otimes \mathcal{H}_P^*$ expressed in the Fermi reference. Once determined the matrix $E^A_{\hat{a}}(s)$ from eqn. (13), one obtain the relation

$$V^{\hat{\alpha}}_{;\hat{\beta}} = E^{\hat{\alpha}}_{\ A} E^B_{\ \hat{\beta}} V^A_{;B}, \tag{18}$$

where $\eta^{\hat{a}\hat{b}} = \eta_{\hat{a}\hat{b}}$.

For the study of the evolution of the interval between the points $P_1(T)$ and $P_2(T)$, with the same value of T, on neighboring curves measured in a Fermi reference, the basic kinematic properties of the congruence of world-lines are the vorticity $\omega_{\hat{\alpha}\hat{\beta}} := h^{\hat{\mu}}_{\hat{\alpha}} h^{\hat{\nu}}_{\hat{\beta}} V_{[\hat{\mu};\hat{\nu}]}$, which represents the angular velocity of the reference frame $\boldsymbol{E}_A$ with respect to the Fermi reference $\boldsymbol{E}_{\hat{\alpha}}$, and the expansion $\theta_{\hat{\alpha}\hat{\beta}} := h^{\hat{\mu}}_{\hat{\alpha}} h^{\hat{\nu}}_{\hat{\beta}} V_{(\hat{\mu};\hat{\nu})}$, which represents the rate of change of the distance between neighboring world-lines. In the particular case considered in this work, these quantities read:

$$\omega_{\hat{\alpha}\hat{\beta}} = -R\varpi(T)\gamma(T)^3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

$$\theta_{\hat{\alpha}\hat{\beta}} = -R^4\alpha\varpi(T)\gamma(T)^5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

where $\gamma(T) := (1 - R^2\varpi^2)^{-1/2}$.

From both the matrix $V^{\hat{\alpha}}_{\ ;\hat{\beta}}$ given in (18), and an initial valued $Y^{\hat{\alpha}}(0)$, it is possible to obtain the separation vector, $\boldsymbol{Y}(s) = Y^{\hat{\alpha}}(s)\,\boldsymbol{E}_{\hat{\alpha}}$, by the integration of (17). Given a value of the parameter s, the distance between two points $\tilde{P}_1, \tilde{P}_2 \in \mathcal{D}$ is the number

$$\text{dist}\,(\tilde{P}_1(s), \tilde{P}_2(s)) := (\eta_{\hat{\alpha}\hat{\beta}} Y^{\hat{\alpha}} Y^{\hat{\beta}})^{1/2}, \tag{21}$$

which in convective representation is equivalent to $\left(h_{AB}(s)Y^A(s)Y^B(s)\right)^{1/2}$, due to the orthogonality condition $\boldsymbol{V} \perp \boldsymbol{Y}$.

4 Integration of the deviation equation of time-like curves

In order to calculate the instantaneous deviation $Y^{\hat{\alpha}}(s)$ of two time-like curves, both corresponding to points on the periphery of the disk, one must solve the linear differential equation (17) in the independent variable T, whose coefficient matrix is given by (18). This matrix depends on the coefficients $E^A_{\hat{\alpha}}$ relating the convective reference $\boldsymbol{E}_A$ and the Fermi reference $\boldsymbol{E}_{\hat{\alpha}}$, which satisfy the differential equation (13). The explicit form of (10) has been obtained using the `tensor package` contained in the symbolic processor MAPLE. Equations (13) and (17) lead to a simultaneous system in coordinate time T of eight first-order differential equations:

$$
\begin{aligned}
\dot{x}_1 &= R\alpha\gamma(T)^2[(-Rx_4x_8 - x_4x_6T + x_3x_7T - x_3x_8\alpha T^2 + x_3x_8\alpha^3T^4R^2 \\
&\quad -x_3x_7T^3R^2\alpha 2)x_2 + (-x_5x_3\alpha T^2 - R^2\alpha^2T^3x_4x_3 \\
&\quad -Rx_4x_5 + x_5x_3\alpha^3T^4R^2)x_1] \\
\dot{x}_2 &= R\alpha\gamma(T)^2[(x_8x_6\alpha^3T^4R^2 - Rx_7x_8 - R^2\alpha^2T^3x_7x_6 - x_8x_6\alpha T^2)x_2 \\
&\quad +(x_4x_6T - x_6x_5\alpha T^2 - Rx_7x_5 - x_3x_7T \\
&\quad -x_6x_4T^3R^2\alpha^2 + x_6x_5\alpha^3T^4R^2)x_1] \\
\dot{x}_3 &= -\alpha TRx_4 + R\alpha^2T^2x_5 \\
\dot{x}_4 &= \frac{\alpha Tx_3}{R} + \alpha x_5 \\
\dot{x}_5 &= \gamma(T)^2(-R\alpha^2T^2x_3 + \alpha R^2x_4) \\
\dot{x}_6 &= -\alpha TRx_7 + R\alpha^2T^2x_8 \\
\dot{x}_7 &= \frac{\alpha Tx_6}{R} + \alpha x_8 \\
\dot{x}_8 &= \gamma(T)^2(R\alpha^2T^2x_6 + \alpha R^2x_7)
\end{aligned}
\tag{22}
$$

for the eight unknowns

$$x_1 = Y^1, x_2 = Y^2, x_3 = E_0^1, x_4 = E_1^1, x_5 = E_2^1, x_6 = E_0^2, x_7 = E_1^2, x_8 = E_2^2$$

The initial values are chosen as follows. Let $\boldsymbol{E}_A(0)$ be the convected reference whose origin is on the initial point $\boldsymbol{X}_1(0)$; the matrix $E_A{}^{\hat{a}}(0)$ is obtained orthonormalizing this basis. On the other hand, to choose an initial value $Y^{\hat{\alpha}}(0)$, we consider, firstly, a tangent vector $\boldsymbol{S} = \ell_0\boldsymbol{E}_2 \in T_{\boldsymbol{X}_1(0)}\mathcal{T}$, where $\ell_0 := R\Delta\Phi$ is small with respect to the radius of the circumference. Using the definition of the separation established in (15), we will take as the initial value $\boldsymbol{Y}(0) = \boldsymbol{h}(0) \cdot \boldsymbol{S}(0)$:

$$\boldsymbol{Y}(0) = \ell_0 E_2^{\hat{\alpha}}(0)\boldsymbol{E}_{\hat{\alpha}}(0), \tag{23}$$

where the relations $\boldsymbol{V} = \boldsymbol{E}_{\hat{0}}$ and $\boldsymbol{E}^{\hat{0}}(\boldsymbol{E}_{\hat{0}}) = -1$ have been taken into account.

The differential system (22) can be solved using numerical techniques. Here we use the Runge-Kutta-Fehlberg (see, e.g., [11]) method applied to a numerical model for the accelerate disk. Here we take $R = 10^{-5}$ as the radius of the disk (we are using geometrical units where $c = 1$). We assume that the disk moves uniformly accelerated from the rest, in $T = 0$, to reach an angular velocity ω at $T_0 = 3 \cdot 10^{-3}$. All points of the periphery of the disk move with the same angular acceleration $\alpha = 0.25 \cdot 10^8$ measured from the reference $\{\boldsymbol{e}_a\}$ as in Grøn's model. We consider the evolution of the points with angular coordinates $\Theta_1 = 0$ and $\Theta_2 = 10^{-6}$. As for the initial values

for the space-like vectors of the Fermi reference we take $\boldsymbol{E}_{\hat{1}}(0) = (0, 1, 0)$ and $\boldsymbol{E}_{\hat{2}}(0) = (0, 0, R^{-1})$.

The numerical integration of the differential system (22), with the considered initial values, has been realized using the function `ode45` implemented in MATLAB, using tolerances `AbsTol` and `RelTol` equal 10^{-8}. The evolution $\boldsymbol{Y}(T)$ of the deviation vector expressed in the Fermi reference $\boldsymbol{E}_{\hat{a}}$ is shown in Fig. 1. This deviation increases with the time in the phase of uniform tangential acceleration
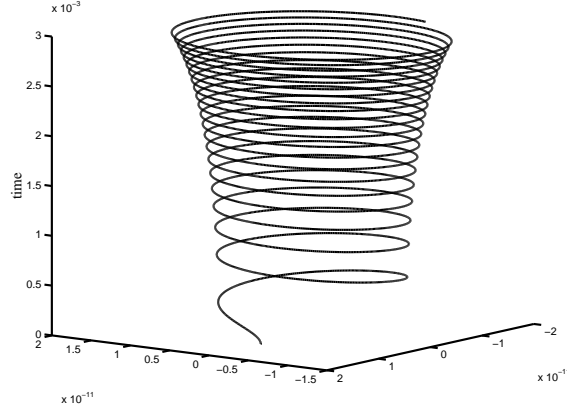

Figure 1: Evolution of the deviation vector between two neighboring worldlines, represented in a Fermi reference: Horizontal axes correspond to the space-like vectors $\boldsymbol{E}_{\hat{\alpha}}$ of the reference frame; the vertical axis represents the time-like vector $\boldsymbol{E}_{\hat{0}}$.

On the other hand, Fig. 2 shows the distance $(\eta_{\hat{\alpha}\hat{\beta}} Y^{\hat{\alpha}} Y^{\hat{\beta}})^{1/2}$ between points $P_1(T)$ and $P_2(T)$ as function of time, obtained from the numerical solution. This figure also shows the corresponding solution obtained by the Grøn method. We observe an complete agreement between the numerical and Grøn solutions. The method we have studied in this work, may be applied to generic flows corresponding to others acceleration programs.
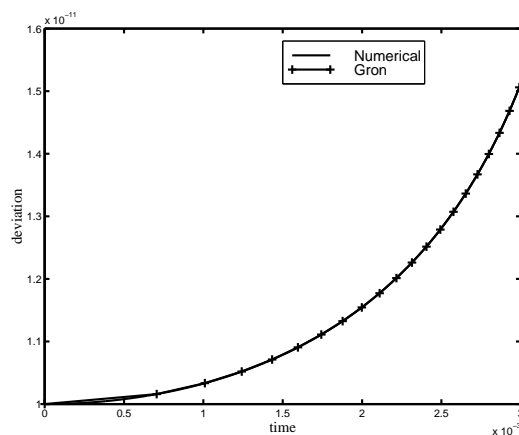
Acknowledgments

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Figure 2: Evolution of the distance between points on neighboring world-lines. The continuous line represents the numerical solution and the line marked with "+" corresponds to the Grøn solution.

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